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Thermal effects in perturbative noncommutative gauge theories

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Abstract: The thermodynamics of gauge theories on the noncommutative plane is studied in perturbation theory. For U(1) noncommutative Yang-Mills we compute the first quantum correction to the ideal gas free energy density and study their behavior in the low and high temperature regimes. Since the noncommutativity scale effectively cutoff interactions at large distances, the theory is regular in the infrared. In the case of $\mathrm{U}(N)$ noncommutative Yang-Mills we evaluate the two-loop free energy density and find that it depends on the noncommutativity parameter through the contribution of non-planar diagrams.

Keywords: Brane Dynamics in Gauge Theories, Gauge Symmetry, 1/N Expansion.

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Contents

1. Introduction

Noncommutative geometry [1] has been a recurrent issue in physics in the last decades. After several attempts to incorporate the mathematical formalism in string field theory [2] and even the standard model [3, 4] and the quantum Hall effect [5], it has recently re-emerged in the context of string/M-theory in the presence of constant background fields [6, 7, 8]. Since the low-energy limit of these configurations is described in terms of a supersymmetric gauge theory living in a noncommutative space, the study of these type of nonlocal field theories has received renewed attention lately. Now, however, because of their stringy connections, new tools are available to study the physics of noncommutative field theories. As a matter of example, the extension of the AdS/CFT correspondence to backgrounds with constant vacuum values of the (Neveu-Schwarz)^2 tensor field [9, 10] makes it feasible the study of noncommutative field theories also in the strong coupling regime [11]–[15]. On the perturbative side, several aspects of noncommutative gauge theories have been recently addressed in [16, 17, 18].

The physical idea behind the application of noncommutative geometry is that of the quantization of space-time itself by introducing noncommuting space-time coordinates

$$[x^\mu, x^\nu] = 2i\theta^{\mu\nu}\,, \qquad \mu,\nu = 0,\ldots,d-1\,. \tag{1.1}$$

Roughly speaking, in ordinary quantum theory the canonical commutation relations lead to a quantization of the phase space that results in a smeared symplectic geometry at short distances due to the uncertainty principle. Following a similar line of reasoning, one is lead to think that the commutation relations (1.1) will smear the space-time picture at distances shorter than $\sqrt{\theta}$, imposing thus a natural cutoff for the description of nature in terms of a local quantum field theory.

In the formalism on noncommutative geometry, the geometrical features of the noncommutative manifold are reconstructed by considering the deformation of the C^*-algebra of continuous complex functions defined on it and vanishing at infinity, using the Weyl product

$$(f \star g)(x) = f(x) e^{i\overleftarrow{\partial}_i \theta^{ij} \overrightarrow{\partial}_j} g(x)\,. \tag{1.2}$$

Consequently, quantum field theories on noncommutative spaces can be formulated by writing the ordinary action and replacing the commutative product with the $\star$-product defined by (1.2); because of its non-polynomial character, the resulting field theories will be non-local. It is actually this non-locality what is argued to smear physics at short distances.

The study of systems at finite temperature usually provides good insights into their physical behavior. In this note we will be concerned with the thermodynamics of Yang-Mills theories on noncommutative spaces (NCYM) of the type $\mathcal{M}_\theta \times \mathbf{R}_t$ where $\mathcal{M}_\theta$ is some $(d-1)$-dimensional noncommutative space, typically $\mathbf{R}_\theta^{d-1}$, characterized by a deformation matrix θ^{ij} $(i,j = 1,\ldots,d-1,\ d>2)$. In this setup, we can compute the thermodynamical potentials using the imaginary time formalism by compactifying the euclidean time to length $\beta = T^{-1}$. The corresponding Feynman rules are thus obtained from the Euclidean Feynman rules of zero temperature noncommutative gauge theories by quantizing the time components of the momenta in units of $2\pi T$. One technical payoff of restricting noncommutativity to the spatial sections is that the resulting non-polynomial functions of the momenta in the Feynman integrals do not involve the discrete Euclidean momentum. Thus, the Matsubara sums that appear in the computation of the free energy are of the same kind that one encounters in ordinary commutative quantum field theories.

In the following we will focus our attention on gauge theories on $\mathbf{R}_\theta^2 \times \mathbf{R}_t$ and $\mathbf{R}_\theta^3 \times \mathbf{R}_t$. Actually, in the four-dimensional case we can always find a rigid orthogonal coordinate transformation $\tilde{x} = Ax$ that takes a generic antisymmetric matrix θ_{ij} to its block off-diagonal form

$$A \begin{pmatrix} 0 & \theta_{12} & \theta_{13} \\ -\theta_{12} & 0 & \theta_{23} \\ -\theta_{13} & -\theta_{23} & 0 \end{pmatrix} A^T = \begin{pmatrix} 0 & \theta & 0 \\ -\theta & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

with $\theta^2 = \theta_{12}^2 + \theta_{13}^2 + \theta_{23}^2$. Thus, when expressed in the appropriate system of coordinates, we see that $\mathbf{R}_\theta^3 \times \mathbf{R}_t$ is actually equivalent to $\mathbf{R}_\theta^2 \times \mathbf{R} \times \mathbf{R}_t$.

The present paper is organized as follows: in section 2 we study the loop corrections to the thermodynamics of U(1) pure NCYM theories at finite temperature. Section 3 will be devoted to the study of the non-abelian U(N) case (and its supersymmetric extensions), where we will find that all the dependence of the two-loop free energy density on θ comes from the contribution of the U(1) part of U(N) to non-planar diagrams. Finally, in section 4 we will summarize our conclusions.

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2. Thermodynamics of U(1) NCYM

Noncommutative pure U(1) Yang-Mills theory is specially interesting, since in this case interaction appears solely as the result of noncommutativity. The action of a U(1) gauge field on $\mathbf{R}_\theta^{d-1} \times \mathbf{R}_t$ can be written as

$$S_{\mathrm{U}(1)} = -\frac{1}{4} \int d^d x \, F_{\mu\nu} \star F^{\mu\nu} \tag{2.1}$$

where the star product is defined by (1.2) and the field strength is given in terms of the Moyal bracket $\{f, g\}_{\mathrm{MB}} = f \star g - g \star f$ as

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ig\{A_\mu, A_\nu\}_{\mathrm{MB}} \,.$$

The Feynman rules for this theory are easily written in momentum space, as shown in references [19, 20]. The resulting diagrammatic expansion is qualitatively similar to ordinary non-abelian Yang-Mills theories, except for the momentum dependence of the vertices through the function $\sin(\theta_{ij} p^i q^j)$. As we will see later, this extra dependence on the momenta with respect to the ordinary non-abelian Yang-Mills theory has important consequences on the infrared behavior of the noncommutative theory.

At one-loop level, the free energy density is determined by the quadratic part of the action and as a consequence it is independent of the noncommutativity of the base space. Thus, the result is identical of that of pure QED$_d$, namely

$$\mathcal{F}(T)_{1-\mathrm{loop}} = -(d-2) \frac{\Gamma(d/2)}{\pi^{d/2}} \zeta(d) T^d \,. \tag{2.2}$$

Corrections to the ideal gas contributions can be computed in perturbation theory using the Feynman rules given in ref. [20] (see also [21]). The first term correcting eq. (2.2) comes from two-loop diagrams. In our case, the final result can be cast in the form

$$\mathcal{F}(T)_{2-\mathrm{loop}} = g^2 (d-2)^2 \int \frac{d^{d-1}p}{(2\pi)^{d-1}} \int \frac{d^{d-1}q}{(2\pi)^{d-1}} \frac{n_b(p) n_b(q)}{\omega_p \omega_q} \sin^2 \theta(p,q) \,, \tag{2.3}$$

where $\omega_p = p$, $\theta(p,q) = \theta_{ij} p^i q^j$ and

$$n_b(p) = \frac{1}{e^{p/T} - 1}$$

is the Bose-Einstein distribution function. Ultraviolet divergences in the zero temperature part of the diagrams are taken care of by inserting the corresponding one-loop couterterms at $T = 0$ [20, 22].

A first thing to be noticed is that, in spite of the similarities between the diagrammatic expansion of the free energy of noncommutative U(1) gauge theory and that of pure YM_d, here the infrared behavior is much softer due to the presence of the factor $\sin^2\theta(p,q)$ which when $p \to 0$ will vanish as $\mathcal{O}(p^2)$. Thus, the two-loop correction (2.3) is well defined for all $d \geq 3$.

Let us first analyze the three dimensional case ($d = 3$). Here, the antisymmetric matrix θ_{ij} can be written as $\theta_{ij} = \theta\epsilon_{ij}$ and the integration over angular variables in (2.3) can be easily performed with the result

$$\mathcal{F}(T)_{2-\mathrm{loop}} = \frac{g^2T^2}{8\pi^2}\int_0^\infty du \int_0^\infty dv \, \frac{1 - J_0(2\theta T^2 uv)}{(e^u - 1)(e^v - 1)} \,. \tag{2.4}$$

We notice that this integral is both infrared ($u, v \to 0$) and ultraviolet ($u, v \to \infty$) convergent. Let us study first the case when the temperature is much smaller than the energy scale $\frac{1}{\sqrt{\theta}}$ associated with noncommutativity effects. If $T\sqrt{\theta} \ll 1$, we can expand the Bessel function in power series and integrate term by term. The result is an asymptotic series valid for small $T\sqrt{\theta}$ whose first term is

$$\mathcal{F}(T)_{2-\mathrm{loop}} \sim \frac{\zeta(3)^2}{2\pi^2} g^2\theta^2T^6 + \cdots$$

The asymptotic character of the series is easily understood by realizing that by expanding the Bessel function in power series and truncating the series we fall short in reproducing the integrand in the ultraviolet region, but this is precisely the region of the integral that is effectively cutoff at low temperatures.

The evaluation of (2.4) in the opposite region $T \gg 1/\sqrt{\theta}$ is more complicated due to the peculiar infrared structure of the theory at hand. We know that, asymptotically, the Bessel function oscillates very fast for large values of the argument. If we introduce a cutoff Λ_θ (that in principle will depend on the value of $T\sqrt{\theta}$) to isolate the infrared sector of the integral, we have that

$$\int_{\Lambda_\theta}^\infty du \int_{\Lambda_\theta}^\infty dv \, \frac{1 - J_0(2\theta T^2 uv)}{(e^u - 1)(e^v - 1)} \sim \int_{\Lambda_\theta}^\infty \frac{du}{e^u - 1} \int_{\Lambda_\theta}^\infty \frac{dv}{e^v - 1} \,,$$

since for large values of the argument the rapidly oscillating Bessel function will be averaged to zero. Thus, when $T\sqrt{\theta} \gg 1$ we can write

$$\mathcal{F}(T)_{2-\mathrm{loop}} \sim \frac{1}{3}\mathcal{F}(T,\Lambda_\theta)^{\mathrm{SU(2)}}_{2-\mathrm{loop}} + f(T,\Lambda_\theta)_{\mathrm{IR}} \,, \tag{2.5}$$

where $\mathcal{F}(T,\Lambda_\theta)^{\mathrm{SU(2)}}_{2-\mathrm{loop}}$ is the two-loop free energy density of ordinary pure YM_3 with gauge group SU(2) and infrared momentum cutoff $\Lambda_\theta T$, and $f(T,\Lambda_\theta)_{\mathrm{IR}}$ is some contribution containing the information from the infrared sector of the theory.

We can interpret the decomposition (2.5) in the sense that, in the large-$T\sqrt{\theta}$ limit, the ultraviolet sector of the U(1) noncommutative gauge theory is well described in terms of an ordinary Yang-Mills theory with $C_2(G) = 2$ [22, 19, 20]. The numerical prefactor 1/3 just reflects the fact that the U(1) noncommutative gauge theory only has one propagating "photon" while the SU(2) Yang-Mills theory has three propagating gluons. Thus, the contribution to the free energy from the ultraviolet part of the theory is given by the free energy *per vector boson* of an SU(2) commutative gauge theory. On the other hand, we see that the theory in the infrared is radically different from ordinary Yang-Mills which is infrared divergent at two loops in three dimensions. This feature is very much reminiscent of the Morita equivalence [23] between U(1) Yang-Mills theory on the noncommutative torus and a U(N) gauge theory on a commutative one in the presence of a magnetic flux, where the infrared sector of the theory is regularized by the presence of the background twisted gauge field [24, 25].

Let us focus our attention now on the four-dimensional case. From the discussion in the Introduction, we know that we can choose coordinates x, y, z such that noncommutativity is restricted to the xy-plane, $[x,y] = 2i\theta$, $[x,z] = [y,z] = 0$. Since the interacting character of the theory is entirely due to the noncommutativity of the base space, we find that our theory will be free whenever the momenta are orthogonal to the xy-plane. Thus, in order to study the integral 2.4 it is convenient to use cylindrical coordinates where the z-coordinate coincides with the "central" direction. Again we can integrate over angular variables to find

$$\begin{aligned}\mathcal{F}(T)_{2-\text{loop}} = & \frac{g^2 T^4}{8\pi^4} \int_{-\infty}^{\infty} du_z \int_{-\infty}^{\infty} dv_z \times \\ & \times \int_0^{\infty} \frac{u\,du}{\sqrt{u^2+u_z^2}} \int_0^{\infty} \frac{v\,dv}{\sqrt{v^2+v_z^2}} \frac{1 - J_0(2T^2\theta\, uv)}{(e^{\sqrt{u^2+u_z^2}}-1)(e^{\sqrt{v^2+v_z^2}}-1)}. \end{aligned} \tag{2.6}$$

As in three dimensions, when $T\sqrt{\theta}$ is small we can expand the Bessel function to get an asymptotic series in powers of $T\sqrt{\theta}$,

$$\mathcal{F}(T)_{2-\text{loop}} \sim 2g^2 \left(\frac{\pi^2}{45}\right)^2 \theta^2 T^8 + \cdots$$

When $T\sqrt{\theta} \gg 1$ it is difficult to estimate the value of the integral (2.6). Since the function resulting from integration of the angular variables is the same than in the three-dimensional case, we can argue along similar lines that, again, the free energy can be decomposed as in (2.5) into a SU(2)-like ordinary Yang-Mills piece and a contribution that takes care of the infrared sector of the theory. Although ordinary Yang-Mills in four-dimensions is infrared finite at two loops and each term in (2.5) is finite in the limit $\Lambda_\theta \to 0$ (when $T\sqrt{\theta} \to \infty$), the infrared part $f(T, \Lambda_\theta)_{\text{IR}}$ gives always a non-trivial contribution, even in this limit. This is due to the fact that in the infrared region the function multiplying $J_0(2T^2\theta uv)$ in the integrand is unbounded and rapidly varying close to $u, v = 0$. Thus, when $T\sqrt{\theta}$ is very large the highly oscillatory Bessel function is not able to average it to zero since the function multiplying it is not approximately constant in a single period.

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In principle, one could compute higher order corrections to the thermodynamic potential (2.6). In ordinary YM_4 at finite temperature the self-interaction of gluons introduce infrared divergences at three loops that do not cancel order by order in perturbation theory and have to be taken care of by resumming the so-called ring diagrams. The result is a mild breakdown of the perturbative expansion that now is no longer a series in g^2 alone but also contains terms of order g^3 [26], $g^4 \log g^2$ [27] and g^5 [28, 29] (for similar results in QED see [30]). The situation is worsened by extra divergences due to the self-interaction of the transverse gluons that invalidates perturbation theory at $\mathcal{O}(g^6)$.

We can study what happens with U(1) Yang-Mills theory on $\mathbf{R}^2_\theta \times \mathbf{R} \times \mathbf{R}_t$ for higher loops contributions to the free energy. A first problem to be solved would be whether the noncommutative U(1) theory itself is renormalizable at $T = 0$ beyond one loop. Let us however assume that the ultraviolet divergences in the zero temperature sector can be handled by some cutoff and concentrate our attention on the (ultraviolet finite) temperature dependent contributions. At three loops, infrared divergences are associated with the existence of a non-vanishing thermal mass at one loop. In the case at hand, however, if we compute the static limit of the one-loop self-energy of the gluon we find that it vanishes quadratically with the external spatial momentum,

$$\Pi_{00}(q_0 = 0, \vec{q} \to 0) = 2\Pi_\mu{}^\mu(q_0 = 0, \vec{q} \to 0) \sim \frac{16\pi^2}{45} g^2 \theta^2 T^4 q_{xy}^2 + \mathcal{O}(q_{xy}^3)\,,$$

where q_{xy} is the modulus of the projection of the external momentum $\vec{q}$ on the xy-plane. The first consequence of this fact, is that three-loop diagrams are free of infrared divergences and therefore the next correction to the free energy is of order g^4.

The soft behavior of U(1) NCYM at low momenta (or large distances) follows from the fact that in this limit the theory becomes free. In physical terms, this is because the theory should reduce itself to its commutative version (a free U(1) pure gauge theory) at length scales much bigger than the typical scale of noncommutative effects, i.e. $\sqrt{\theta}$. In this sense, this scale plays the role of an infrared cutoff for the noncommutativity-induced interactions, which in the ultraviolet resemble those of a non-abelian gauge theory. Thus, the absence of infrared divergences and non-analytic terms in g^2 in the lowest orders in perturbation theory seems to be a generic feature of the whole perturbative expansion which would be a series in integer powers of g^2.

Before closing this section, let us make some remarks about the possible supersymmetric extensions of U(1) NCYM. In the commutative case, we can construct a trivial supersymmetric theory by adding to the pure YM_4 theory the action of a free massless Majorana or Weyl spinor. This theory can be deformed into an interacting supersymmetric theory by switching on the noncommutativity of the base space[1]

$$S=\int d^4x\left[-\frac{1}{4}F_{\mu\nu}\star F^{\mu\nu}+i\bar{\psi}\star\gamma^\mu D_\mu\psi\right], \tag{2.7}$$

where the covariant derivative is defined by $D_\mu\psi=\partial_\mu\psi-i(A_\mu\star\psi-\psi\star A_\mu)$. To compute the two-loop free energy density now we have to add to the result for pure NCYM the contribution coming from the fermion loop with the result

$$\begin{aligned}\mathcal{F}(T,\theta)_{2-\mathrm{loop}}=4g^2\int\frac{d^3p}{(2\pi)^3}\int\frac{d^3q}{(2\pi)^3}\frac{1}{\omega_p\omega_q}\times \\ \times\left[n_b(p)n_b(q)+2n_b(p)n_f(q)+n_f(p)n_f(q)\right]\sin^2\theta(p,q)\,,\end{aligned} \tag{2.8}$$

with $n_f(p)=1/(e^{p/T}+1)$ the Fermi-Dirac distribution function. The structure of expression (2.9) is similar to the two-loop free energy of ordinary SYM [31]. Here again the theory at low momenta becomes trivial (a free gauge field plus an "adjoint" U(1) fermion), rendering the theory infrared finite, while in the ultraviolet it resembles $\mathcal{N}=1$ SYM_4 with $C_2(G)=2$. The three-dimensional case can be worked out along similar lines, starting with the action (2.7) in three dimensions (with ψ a Majorana spinor). As in the non-supersymmetric case, the result is infrared finite.

3. Non-abelian NCYM

Let us consider now non-abelian four-dimensional noncommutative gauge theories on $\mathbf{R}^2_\theta\times\mathbf{R}\times\mathbf{R}_t$. The non-abelian generalization of the action (2.1) is easily written as (group generators are normalized according to $\mathrm{Tr}\,T^aT^b=\frac{1}{2}\delta^{ab}$)

$$S=-\frac{1}{2}\int d^4x\,\mathrm{Tr}[F_{\mu\nu}\star F^{\mu\nu}]\,, \tag{3.1}$$

where $F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu-ig(A_\mu\star A_\nu-A_\nu\star A_\mu)$. Now since our fields are matrix-valued functions, the $\star$-product is defined by tensoring the Weyl product (1.2) with the ordinary product of matrices. A first consequence of this is that in order for the gauge fields to form a closed algebra we should restrict the gauge group to U(N) [8], since we have to demand the group generators to form a closed algebra under ordinary matrix multiplication.[2] From a purely quantum field theoretical

[1]Extended supersymmetric theories can be obtained by considering a similar action in dimension six or ten and performing dimensional reduction to four dimensions.

[2]From the point of view of string theory, one can in principle allow for other groups such as SO(N) and USp($2N$) by introducing orientifold planes. Naively, the introduction of orientifolds will project out the (Neveu-Schwarz)^2 antisymmetric tensor field, and so it looks like one can only obtain commutative space-times. However, the orientation projection is compatible with having quantized background values of the (Neveu-Schwarz)^2 field [32] that will lead to a deformed product of functions on the manifold. This deformed product, however, is not necessarily non-commutative. We thank J. de Boer for a discussion on this point.

point of view, one can try to construct noncommutative gauge theories using Moyal brackets with gauge groups different from U(N). At this level the obstruction to consider these other groups arises in the form of inconsistencies of the resulting non-local quantum field theory. For SU(N), for example, the gauge variation of the vector field A_μ, $\delta A_\mu = \partial_\mu \lambda + i(\lambda \star A_\mu - A_\mu \star \lambda)$, gets an extra piece proportional to the identity matrix

$$\delta A_\mu = \frac{i}{2N}(\lambda^a \star A_\mu^a - A_\mu^a \star \lambda^a)\mathbf{1} + \cdots$$

where the $\star$-product on the right hand side corresponds to that of ordinary functions. Thus, ordinary gauge transformations do not keep the vector field in the adjoint representation of SU(N). Notice, however, that this extra piece scales as $\mathcal{O}(1/N)$ and therefore disappears when $N \to \infty$ where the theory reduces itself to U(N) in the large-N limit, as well as in the commutative limit when we recover ordinary SU(N) YM.

In the ideal gas approximation, the free energy is easily computed by adding the contribution of the different degrees of freedom. Since the noncommutativity parameter θ appears only in the interactions terms, the result is identical to the corresponding ordinary Yang-Mills theory

$$\mathcal{F}(T)_{1-\text{loop}} = -\dim G \, \frac{\pi^2}{45} T^4 \,, \tag{3.2}$$

where $\dim G$ is the dimension of the gauge group, N^2 for U(N).

In order to compute loop corrections to the free gas approximation, we need the Feynman rules for the non-abelian noncommutative Yang-Mills theory. Now, in contrast with the U(1) case, there are interactions surviving the commutative $\theta \to 0$ limit, so the structure of the vertices will be more involved. In order to make the computation more transparent, instead of using the "trigonometric basis" for the gauge group generators [33], we will write Feynman rules using the structure constants f^{abc} and Gell-Mann tensor d^{abc} of the gauge group, defined in terms of the generators T^a by the identities

$$\begin{aligned} f^{abc} &= -2i \, \mathrm{Tr}[T^a, T^b] T^c \,, \\ d^{abc} &= 2 \, \mathrm{Tr}\{T^a, T^b\} T^c \,, \end{aligned}$$

where by $\{\cdot,\cdot\}$ we represent the anticommutator of the two generators.

The Feynman rules for the noncommutative U(N) Yang-Mills can be easily obtained by writing the action (3.1) in momentum space. It turns out that the only change with respect to the Feynman rules of U(N) ordinary Yang-Mills is the replacement on each vertex of the structure constants according to

$$f^{a_1 a_2 a_3} \longrightarrow f^{a_1 a_2 a_3} \cos\theta(p_1, p_2) + d^{a_1 a_2 a_3} \sin\theta(p_1, p_2) \,, \tag{3.3}$$

where a_i is the color index associated with the particle with (incoming) momentum p_i. With this only change, the first quantum correction to (3.2) can be computed to give

$$\mathcal{F}(T,\theta)_{2-\text{loop}} = g^2 \int \frac{d^3p}{(2\pi)^3}\frac{n_b(p)}{\omega_p}\int \frac{d^3q}{(2\pi)^3}\frac{n_b(q)}{\omega_q} \times \\ \times \left[f^{abc}f^{abc}\cos^2\theta(p,q) + d^{abc}d^{abc}\sin^2\theta(p,q)\right]. \qquad (3.4)$$

However, for $\mathrm{U}(N)$ $(N>1)$ it turns out to be that[3]

$$f^{abc}f^{abc} = N(N^2-1), \qquad d^{abc}d^{abc} = N(N^2+1)$$

and therefore the two-loop free energy density is

$$\begin{aligned}\mathcal{F}(T,\theta)_{2-\text{loop}} &= g^2N(N^2-1)\left[\int \frac{d^3p}{(2\pi)^3}\frac{n_b(p)}{\omega_p}\right]^2 + \\ &\quad + 2g^2N\int \frac{d^3p}{(2\pi)^3}\frac{n_b(p)}{\omega_p}\int \frac{d^3q}{(2\pi)^3}\frac{n_b(q)}{\omega_q}\sin^2\theta(p,q) \\ &= \frac{N^2-1}{144}g^2NT^4 + 2g^2N\int \frac{d^3p}{(2\pi)^3}\frac{n_b(p)}{\omega_p}\int \frac{d^3q}{(2\pi)^3}\frac{n_b(q)}{\omega_q}\sin^2\theta(p,q)\,.\end{aligned}$$

The first (θ-independent) part of $\mathcal{F}(T,\theta)_{2-\text{loop}}$ is just the result for the two-loop free energy of ordinary pure $\mathrm{SU}(N)$ Yang-Mills theories in four dimensions [26]. On the other hand, the θ-dependence in the free energy density is subleading in the large-N limit and comes from the contribution of the U(1) part of $\mathrm{U}(N)$ to the two-loop non-planar diagrams. Incidentally, in the large-$T\sqrt{\theta}$ limit the contribution of the U(1) part of $\mathrm{U}(N)$ in the ultraviolet combines with contribution of the $\mathrm{SU}(N)$ part in the same region in such a way that the final result is of order $\mathcal{O}(N^2)$ for finite N. This is a further evidence of the planar character of noncommutative Yang-Mills in the large-θ regime at fixed temperature (or at large temperatures and fixed θ) [34, 12, 15, 16].

In evaluating the free energy density at three loops, we are faced with the customary infrared divergences coming from the "commutative" $\mathrm{SU}(N)$ part, since the U(1) part decouples at large distances ($p \ll 1/\sqrt{\theta}$) and will not induce infrared divergences. As usual, these infrared singularities can be handled by resumming $\mathrm{SU}(N)$ propagators in loops. Since the infrared structure of the theory is similar to commutative $\mathrm{SU}(N)$ Yang-Mills, thermal perturbation theory is expected to break down at order $\mathcal{O}[(g^2N)^3]$. As in the previous section, for small $T\sqrt{\theta}$ we can obtain an asymptotic expansion in powers of $T\sqrt{\theta}$. In the opposite limit we find that the U(1) gauge boson effectively interacts like a non-abelian Yang-Mills field in the ultraviolet.

This result extends to supersymmetric theories straightforwardly [35, 19]. The lagrangean of extended NCSYM theories can be written using the trick of dimensional

[3]For $N=1$ we recover the results of the previous section by setting $f^{aaa}=0$ and $d^{aaa}=2$.

reduction proposed in [31]. We start with a $\mathcal{D}_{\max}$-dimensional "maximal" $\mathcal{N}=1$ theory on $\mathcal{M}_\theta \times \mathbf{R}_t \times (\mathbf{S}^1)_R^{\mathcal{D}_{\max}-4}$ described by

$$\mathcal{L} = -\frac{1}{2}\,\mathrm{Tr}\,F_{\mu\nu} \star F^{\mu\nu} + i\,\mathrm{Tr}\,\bar{\Psi}\Gamma^A \star D_A \Psi\,.$$

As in [31] we take the limit $R \to 0$ and retain only the zero modes of the fields in the internal coordinates, which in loop computations amounts to restrict momenta to four-dimensions in the Feynman integrals. Since all the internal coordinates are commutative, the matrix θ_{ij} is completely oblivious of the fact that we are performing dimensional reduction. Thus, we get $\mathcal{N}=1$, $\mathcal{N}=2$ and $\mathcal{N}=4$ NCSYM$_4$ by taking $\mathcal{D}_{\max} = 4, 6, 10$ respectively.

Again, Feynman rules for U(N) NCSYM are retrieved by the replacement (3.3) on the rules for ordinary SYM. Repeating the analysis above, we find again that all dependence in θ comes from the contribution from U(1) fields in non-planar diagrams, namely

$$\begin{aligned}\mathcal{F}(T,\theta)^{\mathrm{U}(N)}_{2-\mathrm{loop}} = \mathcal{F}(T,\theta=0)^{\mathrm{SU}(N)}_{2-\mathrm{loop}} &+ \frac{1}{2}(\mathcal{D}_{\max}-2)^2 \times \\ &\times g^2 N \int \frac{d^3p}{(2\pi)^3}\int \frac{d^3q}{(2\pi)^3}\frac{1}{\omega_q\omega_p} \times \\ &\times \left[n_b(p)n_b(q) + 2n_b(p)n_f(q) + n_f(p)n_f(q)\right]\sin^2\theta(p,q)\,,\end{aligned}$$

where $\mathcal{F}(T,\theta=0)^{\mathrm{SU}(N)}_{2-\mathrm{loop}}$ is the two-loop free energy density of SU(N) ordinary SYM [36, 31, 37, 38].

4. Conclusions

In the present paper we have studied several aspects of the thermodynamics of perturbative gauge theories on noncommutative spaces. The case of U(1) NCYM is the most interesting example from a dynamical point of view. We have computed the quantum corrections to the one loop result and found that, due to the restoration of the free theory at low momenta, the theory is free of infrared divergences at three loops, due to the fact that the one loop two-point function of the gauge field in the static limit vanishes quadratically with the external momenta. Although we did not explicitly compute corrections beyond order $\mathcal{O}(g^2)$, the general structure of the theory seems to indicate that there is no onset of infrared divergences at any order.

For large values of $T\sqrt{\theta}$ we have seen that the theory behaves in the ultraviolet as an ordinary non-abelian Yang-Mills theory with $C_2(G)=2$. This is in perfect accordance with the result of references [19, 20, 22], in the sense that the ultraviolet divergences of the theory are obtained by averaging the factors $\sin^2\theta(p,q)$ in the amplitudes. On the other hand, the theory is completely different from an ordinary non-abelian gauge theory in the infrared. This is due to the fact that at large distances noncommutativity effects are negligible and a free theory is restored.

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In the case of U(N) NCYM, we have found that the two-loop free energy density only depends on θ through terms of order $\mathcal{O}(1)$ in the large-N limit that correspond to the contribution of the U(1) part of U(N) in non-planar loop diagrams. Since these U(1) contributions are regular in the infrared, we find that the infrared structure of the theory is determined by the SU(N) part and thus it is identical to that of ordinary YM theories. In particular, we will have infrared singularities at three loops that can be resummed to give contributions of order $\mathcal{O}[(g^2N)^{3/2}]$ and $\mathcal{O}[(g^2N)^2 \log g^2N]$, as well as $\mathcal{O}[(g^2N)^{5/2}]$. Thermal perturbation theory will break down at four loops due to the self-interaction of magnetic SU(N) gluons.

Here we have restricted ourselves to gauge theories on the noncommutative plane. It would be interesting to better understand the case of U(1) Yang-Mills theories in the noncommutative torus, specially in the case of rational θ. The computation of quantum corrections in this case is straightforward, and essentially amounts to replacing in our expressions momentum integrals by discrete sums and rescaling $\theta_{ij} \to \pi\theta_{ij}$. By using Morita equivalence, one should be able to relate the free energy of these theories with that of ordinary gauge theories in the presence of twisted background fields.

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